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A Generalization of Favard's Theorem for Polynomials Satisfying a Recurrence Relation

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In this paper, we give the canonical expression for an inner product (defined in $\mathscr{P}$, the linear space of real polynomials), for which the set of orthonormal polynomials satisfies a $(2N+1)$-term recurrence relation. This result is a generalization of Favard's theorem about orthogonal polynomials and three-term recurrence relations. Also, we characterize these inner products in terms of symmetric operators. Similar results are proved for some kinds of discrete Sobolev inner products.

1. INTRODUCTION

We start by recalling some definitions. An infinite symmetric matrix $A=(a_{i,j})_{i,j=0}^{\infty}$ is called positive definite if $\det[(a_{i,j})_{i,j=0}^{n}]>0$ for all $n\geqslant 0$ and non-degenerate if $\det[(a_{i,j})_{i,j=0}^{n}]\neq 0$ for all $n\geqslant 0$. In the same way, a sequence $(a_n)_{n=0}^{\infty}$ is called positive definite or non-degenerate whenever the matrix $(a_{i+j})_{i,j}$ is.

Let B be a real symmetric bilinear form defined on the space of real polynomials $\mathscr{P}$. We recall that the form B is non-degenerate (B generates a pseudo-inner product) if there exists a set of orthogonal polynomials (P_n) with respect to B such that the degree of P_n is n ($\operatorname{dgr}(P_n)=n$); and B is called an inner product if $B(f,f)>0$ for $f\not\equiv 0$. An infinite matrix can be associated to every real symmetric bilinear mapping B putting $a_{n,k}=B(t^n,t^k)$. We can obtain an expression for a set of orthogonal polynomials $(p_n)_n$ with respect to B:

$$p_n(x)=\begin{vmatrix} a_{0,0} & \cdots & a_{0,n} \\ \vdots & \ddots & \vdots \\ a_{n-1,0} & \cdots & a_{n-1,n} \\ 1 & \cdots & x^n \end{vmatrix}, \qquad n\geqslant 0.$$

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If we put $\Delta_n = \det[(a_{i,j})_{i,j=0}^n]$, then $\Delta_{n-1} = \text{coef}_n(p_n)$ (we write $\text{coef}_n(p)$ for the coefficient of t^n in the polynomial p) and $\Delta_{n-1}\Delta_n = B(p_n, p_n)$ (put $\Delta_{-1} = 1$). Hence, if B is an inner product, it is clear that the polynomials

$$q_n = \frac{1}{\sqrt{|\Delta_{n-1}\Delta_n|}} p_n, \qquad n \geq 0$$

are orthonormal, and if B is an pseudo inner product, they are pseudo orthonormal (this means $B(q_n, q_k) = \varepsilon_n \delta_{n,k}$, where $\varepsilon_n = \pm 1$).

It is well-known that another set of orthogonal polynomials $(r_n)_n$ with respect to B must be related to $(p_n)_n$ by the following formula, $r_n = \alpha_n p_n$, where α_n is a real sequence without null terms. From here, it is easy to prove the following characterization:

THEOREM A. (a) *B is a pseudo-inner product if and only if $(a_{i,j})_{i,j}$ is non-degenerate.*

(b) *B is an inner product if and only if $(a_{i,j})_{i,j}$ is positive definite.*

Given a polynomial h, by the operator h we mean the operator defined on $\mathscr{P}$ which results from multiplication by h.

We will use the following well-known results about moment problems:

Given a sequence $(a_n)_n$, there exists a non-discrete positive measure μ (non-discrete means that μ is not a finite combination of Dirac deltas) such that $a_n = \int t^n \, d\mu(t)$ for all $n \geq 0$ if and only if the sequence $(a_n)_n$ is a positive definite one (see [W, pp. 136–138]).

For every real sequence $(a_n)_n$, there exists a function of bounded variation f such that $a_n = \int t^n \, df(t)$ for all $n \geq 0$ (see [Bo]). Actually, the function f can be chosen in a more regular way (see [D1]): For every real sequence $(a_n)_n$, there exists a function f in the Schwartz space S (so a $\mathscr{C}^\infty$ function) vanishing at the negative real numbers and such that $a_n = \int t^n f(t) \, dt$ for all $n \geq 0$. Hence, if we put

$$\mu(t) = \int_{-\infty}^{t} f(x) \, dx,$$

we get a $\mathscr{C}^\infty(\mathbf{R})$ function μ, for which the measure $d\mu(t) = \mu'(t) \, dt = f(t) \, dt$ satisfies $\int t^n \, d\mu(t) = a_n$. Throughout this paper, we often use this fact.

It should be noted that the second and third previous results are also true if the sequence of polynomials $(t^n)_n$ is changed to any sequence of polynomials $(p_n)_n$ with $\text{dgr}(p_n) = n$ (that is if the polynomials $(p_n)_n$ are a basis of $\mathscr{P}$).

The simplest inner or pseudo-inner products are defined by integrating with respect to an arbitrary measure:

$$B(f, g) = \int f(t)\, g(t)\, d\mu(t). \tag{1.1}$$

This kind of inner or pseudo-inner products can be characterized in a simple way:

THEOREM B. *Let B be an inner product, then the following conditions are equivalent*:

(a) *The operator t is symmetric for B, that is $B(tf, g) = B(f, tg)$, for all polynomials f, g.*

(b) *There exists a non-discrete positive measure μ such that B is defined as* (1.1).

Proof. If t is symmetric, then $a_{i,j} = B(t^i, t^j) = B(1, t^{i+j}) = b_{i+j}$. As B is an inner product, the sequence $(b_n)_n$ is positive definite and so there exists a non-discrete positive measure μ such that $b_n = \int t^n\, d\mu(t)$, hence B is defined from μ as (1.1). ∎

An analogous theorem can be stated for pseudo-inner products.

Also, the inner product as (1.1) can be characterized by a certain relation satisfied by its set of orthonormal polynomials: the three-term recurrence relation. Indeed, if the operator t is symmetric for B, it follows easily that, any set of orthonormal polynomials $(q_n)_n$ with respect to B satisfies a three-term recurrence relation,

$$tq_n = a_{n+1} q_{n+1} + b_n q_n + a_n q_{n-1},$$

where $(a_n)_n$ is a real sequence without null terms and $(b_n)_n$ is a real sequence.

The converse is also true, and is known as Favard's Theorem (see [F], [Ch, p. 21]).

THEOREM C (Favard). *Let $(q_n)_n$ be a set of polynomials satisfying the initial conditions $q_0(t) = C \not\equiv 0$, $q_{-1}(t) \equiv 0$ and the following three term recurrence relation*

$$tq_n = a_{n+1} q_{n+1} + b_n q_n + a_n q_{n-1},$$

where $(a_n)_n$ is a real sequence without null terms and $(b_n)_n$ is a real sequence. Then there exists a non-discrete positive measure μ such that $(q_n)_n$ are orthonormal with respect to the inner product defined by μ.

Proof. First of all, it should be noticed that the initial conditions $q_0(t) = C \not\equiv 0$, $q_{-1}(t) \equiv 0$, and the three-term recurrence relation imply that the sequence $(a_n)_n$ must be a nonvanishing one. So this condition could be removed from the hypothesis. This observation was pointed out to the author by F. Marcellán.

From the relation we obtain that $\operatorname{dgr}(q_n) = n$ and so the set of polynomials $(q_n)_n$ is a basis of $\mathscr{P}$. We define an inner product in the following way, if $f = \sum_k a_k q_k$ and $g = \sum_k b_k q_k$, then $B(f, g) = \sum_k a_k b_k$. It is clear that $(q_n)_n$ are orthonormal with respect to B, and from the three term recurrence relation it follows that the operator t is symmetric for B. Now, we can apply Theorem B. ∎

We can state a similar theorem for pseudo-inner products:

THEOREM D. *Let $(q_n)_n$ be a set of polynomials satisfying the initial conditions $q_0(t) = C \not\equiv 0$, $q_{-1}(t) \equiv 0$, and*

$$tq_n = a_{n+1} q_{n+1} + b_n q_n + a_n \varepsilon_n q_{n-1}, \tag{1.2}$$

where $(a_n)_n$ is a real sequence without null terms, $(b_n)_n$ is a real sequence and $(\varepsilon_n)_n$ is a sign sequence (that is $\varepsilon_n = \pm 1$). Then there exists a function μ such that $(q_n)_n$ are pseudo-orthonormal with respect to the pseudo inner product B defined by μ.

Proof. We put $\alpha_n = \varepsilon_0 \varepsilon_1 \cdots \varepsilon_n$, and define $B(f, g) = \sum_k c_k d_k \alpha_k$ if $f = \sum_k c_k q_k$ and $g = \sum_k d_k q_k$. Now, we work as in the previous theorem. ∎

In this paper, we are going to generalize these results. Indeed, we will give an expression for the canonical form of the inner (pseudo-inner) products B such that the operator t^N is symmetric for B. Also, we characterize these inner (pseudo-inner) products giving a relation on its set of orthonormal polynomials: the $(2N + 1)$ term recurrence relation. In order to illustrate it, we state the result for $N = 2$.

THEOREM 1. *Let B be a real symmetric bilinear form, then the following conditions are equivalent:*

(a) *The operator t^2 is symmetric for B, that is $B(t^2 f, g) = B(f, t^2 g)$, for all polynomials f, g.*

(b) *There exist two functions μ and ν such that B is defined as follows*

$$B(f, g) = \int f(t)\, g(t)\, d\mu(t) + \int (f(t) - f(-t))(g(t) - g(-t))\, d\nu(t). \tag{1.3}$$

Moreover, if we put $\alpha_n = \int t^n \, d\mu(t)$ *and* $M_n = 4 \int t^n \, dv(t)$, *then the matrix*

$$a_{n,k} = \begin{cases} \alpha_{n+k}, & \text{if } n \text{ or } k \text{ are even}, \\ \alpha_{n+k} + M_{n+k}, & \text{if } n \text{ and } k \text{ are odd}. \end{cases}$$

is positive definite (non-degenerate) if and only if B is an inner product (pseudo-inner product).

In this case, the set of orthonormal polynomials with respect to an inner product like (1.3), satisfies the five-term recurrence relation

$$t^2 q_n = a_{n+2} q_{n+2} + b_{n+1} q_{n+1} + c_n q_n + b_n q_{n-1} + a_n q_{n-2}, \tag{1.4}$$

where $(a_n)_n$ is a real sequence without null terms and $(b_n)_n$, $(c_n)_n$ are real sequences. Also, we get the generalization of Favard's Theorem:

THEOREM 2. *Let* $(q_n)_n$ *be a set of polynomials satisfying the initial conditions* $q_0(t) = C \not\equiv 0$, $q_{-1}(t)$, $q_{-2}(t) \equiv 0$, *and the five-term recurrence relation* (1.4). *Then there exist two functions* μ *and* v *such that the bilinear form*

$$B(f, g) = \int f(t)\, g(t)\, d\mu(t) + \int (f(t) - f(-t))(g(t) - g(-t))\, dv(t)$$

is an inner product and the polynomials $(q_n)_n$ *are orthonormal with respect to B.*

Notice that, in this case, the theorem does not guarantee the positivity of the measures μ and v. We give some examples proving that, although B is an inner product, these measures can not be chosen to be positive. In a subsequent paper ([D2]) some positivity conditions on the measure will be given in order to extend these inner products from the linear space of polynomials to an associated Hilbert space.

All these results will be extended for real symmetric bilinear forms B, such that the operator h (here, h is a fixed polynomial) is symmetric for B.

Finally, we give a characterization in terms of symmetric operators for some classes of discrete Sobolev inner products (see Section 3). Orthonormal polynomials with respect to these inner products have extensively been studied the last few years (see [BM1, BM2, MR, K, MV], ...).

2. THE $(2N+1)$-TERM RECURRENCE RELATION

In this section, we are going to extend Theorems B, C, and D for $N > 1$. In order to show the expression for the canonical form of the real symmetric bilinear mappings B such that the operator t^N is symmetric for B, we need the following definition: let N be a non-negative integer. Put w for a primitive Nth root of the unity. Given a non-negative integer m such that $0 \leqslant m \leqslant N-1$, we define the operator

$$T_{m,N} : \mathscr{P} \to \mathscr{P}$$

as follows:

$$T_{m,N}(f)(t) = \frac{1}{N} \sum_{k=0}^{N-1} (w^{-m})^k f(w^k t).$$

Note that if $f = \sum_i a_i t^i$, then

$$T_{m,N}(f)(t) = \sum_i a_{iN+m} t^{iN+m}. \tag{2.1}$$

For $m=1$ and $N=2$, we get $T_{1,2}(f)(t) = \frac{1}{2}(f(t) - f(-t))$.

Hence, we state the following:

THEOREM 3. *Let B a real symmetric bilinear form, then the following are equivalent*:

(a) *The operator t^N is symmetric for B, that is $B(t^N f, g) = B(f, t^N g)$, for all polynomials f, g.*

(b) *There exist functions μ_0 and $\mu_{m,m'}$ for $1 \leqslant m, m' \leqslant N-1$, with $\mu_{m,m'} = \mu_{m',m}$, such that B is defined as follows*

$$B(f, g) = \int f(t)\, g(t)\, d\mu_0(t) + \sum_{1 \leqslant m, m' \leqslant N-1} \int T_{m,N}(f)(t)\, T_{m',N}(g)(t)\, d\mu_{m,m'}(t).$$

(c) *There exist functions $\mu_{m,m'}$ for $0 \leqslant m, m' \leqslant N-1$, with $\mu_{m,m'} = \mu_{m',m}$, such that B is defined as follows*:

$$B(f, g) = \sum_{0 \leqslant m, m' \leqslant N-1} \int T_{m,N}(f)(t)\, T_{m',N}(g)(t)\, d\mu_{m,m'}(t).$$

Proof. In order to prove (a) → (b), put $\alpha_n = B(1, t^n)$ and, for $1 \leqslant m$, $m' \leqslant N-1$ consider the sequences

$$\beta_n^{(m,m')} = B(t^m, t^{nN+m'}), \qquad n \geqslant 0, \tag{2.2}$$

and

$$M_n^{(m,m')} = \beta_n^{(m,m')} - \alpha_{nN+m+m'}, \qquad n \geq 0. \tag{2.3}$$

Note that, since t^N is symmetric for B, it follows that $\beta_n^{(m,m')} = \beta_n^{(m',m)}$ and $M_n^{(m,m')} = M_n^{(m',m)}$. We choose the function μ_0 such that $\int t^n \, d\mu_0(t) = \alpha_n$ and the functions $\mu_{m,m'}$ such that $\mu_{m,m'} = \mu_{m',m}$, and

$$\int t^{nN+m+m'} \, d\mu_{m,m'}(t) = M_n^{(m,m')} \qquad \text{for} \quad n \geq 0. \tag{2.4}$$

Again, since the operator t^N is symmetric for B, it follows that if $i = kN$ or $j = k'N$, then

$$B(t^i, t^j) = B(1, t^{i+j}) = \alpha_{i+j}, \tag{2.5}$$

and if $1 \leq m, m' \leq N-1$, $i = kN+m$ and $j = k'N+m'$, then

$$\begin{aligned} B(t^i, t^j) &= B(t^{kN+m}, t^{k'N+m'}) \\ &= B(t^m, t^{(k+k')N+m'}) = \beta_{k+k'}^{(m,m')}. \end{aligned} \tag{2.6}$$

Hence, if $1 \leq m, m' \leq N-1$ and $f = \sum_i c_i t^i$, $g = \sum_j d_j t^j$ are two polynomials, from (2.1) and (2.4), we have

$$\begin{aligned} &\int T_{m,N}(f)(t) \, T_{m',N}(g)(t) \, d\mu_{m,m'}(t) \\ &\quad = \sum_{i,j} c_{iN+m} d_{jN+m'} \int t^{iN+jN+m+m'} \, d\mu_{m,m'}(t) \\ &\quad = \sum_{i,j} c_{iN+m} d_{jN+m'} M_{i+j}^{(m,m')}. \end{aligned} \tag{2.7}$$

From (2.5), (2.6), (2.3), and (2.7), we get

$$\begin{aligned} B(f, g) &= \sum_{i,j} c_i d_j B(t^i, t^j) \\ &= \sum_{i,j} c_{iN} d_j B(t^{iN}, t^j) + \sum_{i,j} c_i d_{jN} B(t^i, t^{jN}) - \sum_{i,j} c_{iN} d_{jN} B(t^{iN}, t^{jN}) \\ &\quad + \sum_{1 \leq m,m' \leq N-1} \sum_{i,j} c_{iN+m} d_{jN+m'} B(t^{iN+m}, t^{jN+m'}) \\ &= \sum_{i,j} c_{iN} d_j \alpha_{iN+j} + \sum_{i,j} c_i d_{jN} \alpha_{i+jN} - \sum_{i,j} c_{iN} d_{jN} \alpha_{(i+j)N} \\ &\quad + \sum_{1 \leq m,m' \leq N-1} \sum_{i,j} c_{iN+m} d_{jN+m'} \beta_{i+j}^{(m,m')} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j} c_{iN} d_j \alpha_{iN+j} + \sum_{i,j} c_i d_{jN} \alpha_{i+jN} - \sum_{i,j} c_{iN} d_{jN} \alpha_{(i+j)N} \\
&\quad + \sum_{1 \leqslant m,m' \leqslant N-1} \sum_{i,j} c_{iN+m} d_{jN+m'} \alpha_{iN+jN+m+m'} \\
&\quad + \sum_{1 \leqslant m,m' \leqslant N-1} \sum_{i,j} c_{iN+m} d_{jN+m'} M_{i+j}^{(m,m')} \\
&= \sum_{i,j} c_i d_j \alpha_{i+j} + \sum_{1 \leqslant m,m' \leqslant N-1} \int T_{m,N}(f)(t)\, T_{m',N}(g)(t)\, d\mu_{m,m'}(t) \\
&= \int f(t)\, g(t)\, d\mu_0(t) \\
&\quad + \sum_{1 \leqslant m,m' \leqslant N-1} \int T_{m,N}(f)(t)\, T_{m',N}(g)(t)\, d\mu_{m,m'}(t).
\end{aligned}
$$

Now, we are going to prove (b) $\to$ (c). Indeed, consider the functions v_l with $0 \leqslant l \leqslant N-1$ such that if $n \geqslant 0$ and $0 \leqslant m \leqslant N-1$ then

$$
\int t^{nN+m}\, dv_l = \begin{cases} 0, & \text{for } m \neq l, \\ \int t^{nN+l}\, d\mu_0, & \text{for } m = l. \end{cases}
$$

with this choice, we have $d\mu_0 = \sum_{l=0}^{N-1} dv_l$. If f is a polynomial then $f(t) = \sum_{m=0}^{N-1} T_{m,N}(f)(t)$, hence

$$
\int f(t)\, g(t)\, d\mu_0(t) = \sum_{l=0}^{N-1} \sum_{i=0}^{1} \sum_{m+m'=iN+l} \int T_{m,N}(f)(t)\, T_{m',N}(g)(t)\, dv_l,
$$

and so, Condition (c) follows.

Since $T_{m,N}(t^N f)(t) = t^N T_{m,N}(f)(t)$ for $0 \leqslant m \leqslant N-1$, (c) $\to$ (a) follows. ∎

It should be noticed that the operators $T_{m,N}$ which appear in the previous theorem can be changed to the operators $R_{m,N} 0 \leqslant m \leqslant N-1$ defined by

$$
R_{m,N}(f)(t) = \frac{1}{Nt^m} \sum_{k=0}^{N-1} (w^{-m})^k f(w^k t) = \sum_i a_{iN+m} t^{iN},
$$

for any polynomial $f = \sum_i a_i t^i$.

Putting $N = 2$ in Theorem 3, we obtain Theorem 1 as a corollary of Theorem 3.

If we put $\mu_{m,m'} = 0$ when $m \neq m'$ and

$$
\int t^n\, d\mu_{m,m} = \begin{cases} 0, & \text{if } n \neq 2m, \\ (m!)^2 M_m, & \text{if } n = 2m, \end{cases}
$$

in the canonical expression given in previous Theorem, we get the well-known inner products

$$B(f, g) = \int f(t)\, g(t)\, d\mu(t) + \sum_{m=0}^{N-1} M_m f^{(m)}(0)\, g^{(m)}(0).$$

Note that these Sobolev type inner products are not characterized by the condition that t^N is symmetric for B. In Section 3, we get a complete characterization of them in terms of symmetric operators.

Now, we are going to extend Favard's Theorem. Given N a positive integer, and a set of polynomials $(p_n)_n$ with $p_0(t)$ a constant different from 0, we say that the polynomials $(p_n)_n$ satisfy a $(2N+1)$-term recurrence relation if the formula

$$t^N p_n(t) = c_{n,0}\, p_n(t) + \sum_{l=1}^{N} (c_{n,l}\, p_{n-l}(t) + c_{n+l,l}\, p_{n+l}(t)) \tag{2.8}$$

holds, where $(c_{n,N})_n$ is a real sequence without null terms and $(c_{n,l})_n$ with $0 \leqslant l \leqslant N-1$ are real sequences (of course, if $l < 0$ then $c_{n,l} = p_l = 0$).

Note that for $N = 1$, we obtain the three-term recurrence relation for orthonormal polynomials, and for $N = 2$, we obtain the five-term recurrence relation (1.4).

Also, it will be interesting to consider $(2N+1)$-term recurrence relation like

$$t^N p_n(t) = c_{n,0}\, p_n(t) + \sum_{l=1}^{N} ((\varepsilon_n \varepsilon_{n-l})\, c_{n,l}\, p_{n-l}(t) + c_{n+l,l}\, p_{n+l}(t)), \tag{2.9}$$

where $(c_{n,N})_n$ is a real sequence without non-null terms, $(c_{n,l})_n$ with $0 \leqslant l \leqslant N-1$ are real sequences and $(\varepsilon_n)_n$ is a sequence of signs (that is $\varepsilon_n = \pm 1$).

To begin with, it is easy to prove that if the operator t^N is symmetric for an inner (pseudo-inner) product B, then the set of orthonormal (pseudo-orthonormal) polynomials with respect to B satisfies a $(2N+1)$-term recurrence relation like (2.8) (or (2.9)).

The converse result is also true, and is the generalization of Favard's Theorem:

THEOREM 4. *Let N be a positive integer and $(p_n)_n$ a set of polynomials satisfying a $(2N+1)$-term recurrence relation as* (2.8) (*or* (2.9)). *Then there exist functions μ_0 and $\mu_{m,m'}$ for $1 \leqslant m, m' \leqslant N-1$, with $\mu_{m,m'} = \mu_{m',m}$, such* *that the polynomials $(p_n)_n$ are orthonormal with respect to the inner product (pseudo-inner product) B defined by*

$$B(f, g) = \int f(t)\, g(t)\, d\mu_0(t) + \sum_{1 \leqslant m, m' \leqslant N-1} \int T_{m,N}(f)(t)\, T_{m',N}(g)(t)\, d\mu_{m,m'}(t).$$

Proof. The proof is the same as the one for $N = 1$. Indeed, from the relation we obtain that $dgr(p_n) = n$ and so the set of polynomials $(p_n)_n$ is a basis in $\mathscr{P}$. We define an inner product (pseudo-inner product) in the following way: if $f = \sum_k a_k p_k$ and $g = \sum_k b_k p_k$ then $B(f, g) = \sum_k a_k b_k$ $(B_1(f, g) = \sum_k a_k b_k \varepsilon_k$ if $(p_n)_n$ satisfy the pseudo-relation). It is clear that $(p_n)_n$ are orthonormal (pseudo-orthonormal) with respect to B (B_1), and from the $(2N+1)$-term recurrence relation it follows that the operator t^N is symmetric for B (B_1). Now, we apply Theorem 3. ∎

EXAMPLES. (I) Let us consider two positive measures ρ_1, ρ_2 with support contained in $[0, +\infty)$, and their associated orthonormal polynomials $(r_n)_n, (s_n)_n$ respectively. We define the following sequence of polynomials

$$p_{2n}(t) = r_n(t^2), \qquad p_{2n+1} = t s_n(t^2), \qquad n \geqslant 0.$$

Straightforward computations show that the sequence of polynomials $(p_n)_n$ satisfies a five-term recurrence relation (see (1.4)), with the following coefficients:

$$a_n = \begin{cases} a^r_k, & \text{if } n = 2k, \\ a^s_k, & \text{if } n = 2k+1, \end{cases} \quad b_n = 0, \quad c_n = \begin{cases} b^r_k, & \text{if } n = 2k, \\ b^s_k, & \text{if } n = 2k+1, \end{cases} \quad n \geqslant 0,$$

where $(a^r_n)_n, (b^r_n)_n, (a^s_n)_n, (b^s_n)_n$ are the coefficients in the three-term recurrence relations of the polynomials $(r_n)_n, (s_n)_n$, respectively.

So, the polynomials $(p_n)_n$ are orthonormal with respect to an inner product like (1.3) or like that of part (c) in Theorem 3. Now, we give an expression (in terms of ρ_1, ρ_2) for the measures which appear in these inner products. Let us consider the following even positive measures

$$\mu(A) = \rho_1\{t^2 : t \in A, t \geqslant 0\} + \rho_1\{t^2 : t \in A, t \leqslant 0\},$$

$$\sigma(A) = \rho_2\{t^2 : t \in A, t \geqslant 0\} + \rho_2\{t^2 : t \in A, t \leqslant 0\},$$

and the measure (it is likely a non-positive measure)

$$\nu = \sigma - t^2 \mu$$

($t^2\mu$ is the measure with density t^2 with respect to μ). Then, we have

$$\int_0^{\infty} f(t^2)\,d\mu = \int_0^{+\infty} f(t)\,d\rho_1, \qquad \int_0^{\infty} f(t^2)\,d\sigma = \int_0^{+\infty} f(t)\,d\rho_2,$$

$$\int_0^{\infty} f(t^2)\,d\nu = \int_0^{+\infty} f(t)\,d\rho_2 - \int_0^{+\infty} tf(t)\,d\rho_1.$$

From these formulas, it follows that the polynomials $(p_n)_n$ are orthonormal with respect to the inner products

$$B(f,g) = \frac{1}{4}\int_0^{\infty} (f(t)+f(-t))(g(t)+g(-t))\,d\mu + \frac{1}{4}\int_0^{\infty} \left(\frac{f(t)-f(-t)}{t}\right)\left(\frac{g(t)-g(-t)}{t}\right) d\sigma$$

and

$$B(f,g) = \frac{1}{2}\int_{-\infty}^{+\infty} f(t)\,g(t)\,d\mu + \frac{1}{4}\int_0^{\infty} \left(\frac{f(t)-f(-t)}{t}\right)\left(\frac{g(t)-g(-t)}{t}\right) d\nu.$$

The converse of this example is true, that is, if a sequence of polynomials $(p_n)_n$ satisfying the initial condition $p_{-1}(t) = p-2(t) = 0$, $p_0(t) = C_0 \not\equiv 0$, $p_1(t) = C_1 t \not\equiv 0$ and a five-term recurrence relation (1.4) with $b_n = 0$ for all $n \geqslant 0$ then they are orthonormal with respect to an inner product like the first above written where μ and σ are positive measures.

(I.1) Taking $r_n(t) = L_n^{\alpha}(t)$, $s_n(t) = L_n^{\beta}(t)$ $(\alpha > -1, \beta > 0)$ (as usual $(L_n^{\alpha}(t))_n$ are the Laguerre polynomials), we get a generalization of the Hermite polynomials

$$H^{\alpha,\beta}(t)_n = \begin{cases} (-1)^k\, 2^{2k} k!\, L_k^{\alpha}(t^2), & \text{for } n = 2k, \\ (-1)^k\, 2^{2k+1} k!\, tL_k^{\beta}(t^2), & \text{for } n = 2k+1, \end{cases}$$

which are orthogonal which respect to the inner product

$$B(f,g) = \int_{-\infty}^{+\infty} f(t)\,g(t)\,|t|^{2\alpha+1} e^{-t^2}\,dt + \tfrac{1}{2}\int_0^{\infty} (f(t)-f(-t))(g(t)-g(-t))(t^{2\beta-1} - t^{2\alpha+1})\,e^{-t^2}\,dt.$$

The generating function

$$(1+2tw(1+4w^2)^{\alpha-\beta})(1+4w^2)^{-\alpha-1}e^{4t^2w^2(1+4w^2)^{-1}}=\sum_{n=0}^{\infty}H_n^{\alpha,\beta}(t)\frac{w^n}{[n/2]!}$$

can be derived from the definition of these generalized Hermite polynomials and the generating function for the Laguerre polynomials. For $-\alpha=\beta=\frac{1}{2}$, we have the classical Hermite polynomials, and for $\alpha=\mu-\frac{1}{2}=\beta-1$, we have the generalized Hermite polynomials which were introduced by Szegő (see also [Ch, p. 156]).

(II) As we wrote in the Introduction of this paper, Theorem 4 does not guarantee the positivity of the measures μ_0 or $\mu_{m,m'}$. We show an example for $N=2$ which shows that, in fact, although B is an inner product, these measures cannot be chosen to be positive.

Consider a positive definite sequence $(\alpha_n)_n$ such that the sequence defined by

$$\gamma_n=\begin{cases}\alpha_n, & \text{if} \quad n=4k,\\ 0, & \text{otherwise,}\end{cases}$$

is not a positive definite one (for instance, we can take $\alpha_n=1/(n+1)$). Put

$$\beta_n=\begin{cases}-\alpha_n, & \text{if} \quad n=4k,\\ \alpha_n, & \text{if} \quad n=4k+2,\\ 0, & \text{otherwise.}\end{cases}$$

Let ρ be a positive measure such that $\int t^n\,d\rho=\alpha_n$, μ a function such that $\int t^n\,d\mu=\gamma_n$ and ν a function such that $\int t^n\,d\nu=\beta_n$. Put

$$B(f,g)=\int f(t)\,g(t)\,d\mu(t)+\tfrac{1}{4}\int(f(t)-f(-t))(g(t)-g(-t))\,d\nu(t). \tag{2.10}$$

First of all, suppose that there exist two positive measures μ', ν' such that

$$B(f,g)=\int f(t)\,g(t)\,d\mu'(t)+\int(f(t)-f(-t))(g(t)-g(-t))\,d\nu'(t). \tag{2.11}$$

From (2.10) and (2.11), it follows that $\gamma_n=B(1,t^n)=\int t^n\,d\mu'$, so $(\gamma_n)_n$ should be a positive definite sequence, but by construction, this is not true.

Now, we are going to prove that B is an inner product. Indeed, if $f=\sum_k c_k t^k$ is a polynomial, we write $f=f_0+f_1+f_2+f_3$, where

$f_i = \sum_k c_{4k+i} t^{4k+i}$, $i = 0, 1, 2, 3$. From the above construction, we get: if $i + j = 4k$ for some k, then

$$\int f_i(t) f_j(t)\, d\mu(t) = \int f_i(t) f_j(t)\, d\rho(t); \tag{2.12}$$

if $i + j \neq 4k$, for any k, then

$$\int f_i(t) f_j(t)\, d\mu(t) = 0; \tag{2.13}$$

if $i + j = 4k + 2$ for some k, then

$$\int f_i(t) f_j(t)\, d\nu(t) = \int f_i(t) f_j(t)\, d\rho(t); \tag{2.14}$$

if $i + j \neq 2k$ for any k, then

$$\int f_i(t) f_j(t)\, d\nu(t) = 0; \tag{2.15}$$

and, if $i + j = 4k$ for some k, then

$$\int f_i(t) f_j(t)\, d\nu(t) = -\int f_i(t) f_j(t)\, d\mu(t). \tag{2.16}$$

Hence, if $f \neq 0$, from (2.10), (2.12), (2.13), (2.14), and (2.15), we have (note that ρ is a positive measure):

$$\begin{aligned} B(f, f) &= B(f_0 + f_1 + f_2 + f_3, f_0 + f_1 + f_2 + f_3) \\ &= \int f_0(t) f_0(t)\, d\mu(t) + \int f_2(t) f_2(t)\, d\mu(t) \\ &\quad + 2 \int f_1(t) f_3(t)\, d\mu(t) + 2 \int f_1(t) f_3(t)\, d\nu(t) \\ &\quad + \int f_1(t) f_1(t)\, d\nu(t) + \int f_3(t) f_3(t)\, d\nu(t) \\ &= \sum_{k=0}^{3} \int f_k^2(t)\, d\rho > 0. \end{aligned}$$

So, B is an inner product.

(III) In the previous example, the measure μ cannot be chosen to be positive. Here, we show some examples where the measure μ is positive,

B is an inner product, but the measure ν cannot be chosen to be positive. The next lemma will be the key to show these examples, although it is interesting in itself:

LEMMA 1. *Let T be an infinite real matrix, $T=(t_{i,j})_{i,j=0}^{\infty}$. For a sequence of real numbers $(a_n)_n$, let us consider the following two sequences of numbers $(\alpha_n)_n$ and $(\beta_n)_n$:*

$$\alpha_n = \begin{vmatrix} a_0 & \cdots & a_n \\ a_1 & \cdots & a_{n+1} \\ \vdots & \ddots & \vdots \\ a_{n-1} & \cdots & a_{2n-1} \\ a_n & \cdots & a_{2n} \end{vmatrix}, \qquad n \geqslant 0, \tag{2.17}$$

and

$$\beta_n = \begin{vmatrix} a_0 + t_{0,0} & \cdots & a_n + t_{0,n} \\ a_1 + t_{1,0} & \cdots & a_{n+1} + t_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-1} + t_{n-1,0} & \cdots & a_{2n-1} + t_{n-1,n} \\ a_n + t_{n,0} & \cdots & a_{2n} + t_{n,n} \end{vmatrix}, \qquad n \geqslant 0. \tag{2.18}$$

Then, for every finite matrix T there exists a sequence $(a_n)_n$ such that $\alpha_n, \beta_n > 0$ for all $n \geqslant 0$.

Proof. Indeed, expanding the determinants which define the sequences $(\alpha_n)_n$, $(\beta_n)_n$ by the last row, we get:

$$\alpha_n = \alpha_{n-1} a_{2n} + A_n,$$
$$\beta_n = \beta_{n-1}(a_{2n} + t_{n,n}) + B_n.$$

It should be noted that the numbers $\alpha_{n-1}, A_n, \beta_{n-1}$ and B_n depend on $a_0, a_1, ..., a_{2n-1}$ and the matrix T, but do not depend on a_{2n}. Hence, we chose $a_0 > |t_{0,0}|$, $a_{2n+1} = 0$ and by recurrence

$$a_{2n} > \max\left\{\frac{-B_n - \beta_{n-1} t_{n,n}}{\beta_{n-1}}, \frac{-A_n}{\alpha_{n-1}}\right\}.$$

From this choice, the lemma follows. ∎

This lemma says that given an infinite real matrix T, there exists a positive definite Hankel matrix such that if we modify the Hankel matrix by summing the matrix T, the modified matrix which is obtained is also positive definite. Now, in an inner product defined by

$$B(f, g) = \int f(t)\, g(t)\, d\mu(t) + \int (f(t) - f(-t))(g(t) - g(-t))\, d\nu(t),$$

we modify the Hankel matrix of μ by summing the symmetric real matrix $T = (t_{i,j})_{i,j=0}^{\infty}$, defined by

$$t_{i,j} = \begin{cases} 4 \int t^i t^j\, d\nu(t), & \text{if } i, j \text{ are odd,} \\ 0, & \text{if not.} \end{cases}$$

From Lemma 1, we can get examples of inner product B like (1.4) such that μ is a positive measure but ν cannot be chosen to be positive.

In general, given a natural number N and for $1 \leqslant m, m' \leqslant N-1$ a function $\mu_{m,m'}$ with $\mu_{m,m'} = \mu_{m',m}$, using Lemma 1, we can find a positive measure μ such that the symmetric bilinear form defined by

$$B(f, g) = \int f(t)\, g(t)\, d\mu_0(t)$$
$$+ \sum_{1 \leqslant m, m' \leqslant N-1} \int T_{m,N}(f)(t)\, T_{m',N}(g)(t)\, d\mu_{m,m'}(t)$$

is an inner product, although $\mu_{m,m'}$ are not positive measures.

In the rest of this Section we will obtain the canonical form of the real symmetric bilinear mappings B for which the operator h (here h is a fixed polynomial) is symmetric for B.

Let h be a fixed polynomial, and $N = \operatorname{dgr}(h)$. It is clear that the following set of polynomials is a basis of $\mathscr{P}$:

$$\mathscr{B}_h = \{1, t, ..., t^{N-1}, h, th, ..., t^{N-1}h, h^2, th^2, ...\}$$
$$= \{t^m h^k, k \geqslant 0, 0 \leqslant m \leqslant N-1\}.$$

If $0 \leqslant m \leqslant N-1$, we are going to define the operators $T_{m,h}$ related to h, in the same way as the operators $T_{m,N}$ for t^N. Indeed, for a polynomial f, we can write $f = \sum_{k \geqslant 0, 0 \leqslant n \leqslant N-1} a_{n,k} t^n h^k$. Then we define

$$T_{m,h}(f)(t) = \sum_{k \geqslant 0} a_{m,k} t^m h^k. \tag{2.19}$$

Notice that, for $h(t) = t^N$, the operator T_{m,t^N} defined by (2.19) is the operator $T_{m,N}$.

Then, we obtain the following extension of Theorem 3.

THEOREM 5. *Let B a real symmetric bilinear form, then the following are equivalent*:

(a) *The operator h is symmetric for B, that is $B(hf, g) = B(f, hg)$, for all polynomials f, g.*

(b) *There exist functions $\mu_{m,m'}$ for $0 \leqslant m \leqslant m' \leqslant N-1$, such that B is defined as follows*:

$$B(f, g) = \sum_{0 \leqslant m \leqslant m' \leqslant N-1} \int T_{m,h}(f)(t)\, T_{m',h}(g)(t)\, d\mu_{m,m'}(t).$$

(c) *There exist functions μ_0 and $\mu_{m,m'}$ for $1 \leqslant m \leqslant m' \leqslant N-1$, such that B is defined as follows*:

$$B(f, g) = \int f(t)\, g(t)\, d\mu_0 + \sum_{1 \leqslant m \leqslant m' \leqslant N-1} \int T_{m,h}(f)(t)\, T_{m',h}(g)(t)\, d\mu_{m,m'}(t).$$

Proof. In order to prove (a) $\to$ (b), we consider the following bases of $\mathscr{P}$, $(t^n)_n$ and $\mathscr{B}_h$ (the latter is defined above), and the linear mapping $R: \mathscr{P} \to \mathscr{P}$ defined by $R(t^{m+kN}) = t^m h^k$, i.e., the mapping for the change of basis. It is easy to prove that $R(t^N f) = hR(f)$ for any polynomial f. So the operator t^N is symmetric for the real symmetric bilinear mapping B', defined by $B'(f, g) = B(R(f), R(g))$. Hence, from part (c) of Theorem 3, we have functions $\nu_{m,m'}$, with $0 \leqslant m, m' \leqslant N-1$ such that

$$B'(f, g) = \sum_{0 \leqslant m \leqslant m' \leqslant N-1} \int T_{m,N}(f)(t)\, T_{m',N}(g)(t)\, d\nu_{m,m'}(t). \tag{2.20}$$

Now, fixing $0 \leqslant m, m' \leqslant N-1$, since the degree of the polynomials $t^{m+m'}h^k$ is different for different values of k, there exists a function $\mu_{m,m'}$ such that

$$\int t^{m+m'} h^k\, d\mu_{m,m'} = \int t^{kN+m+m'}\, d\nu_{m,m'}.$$

Hence, we get

$$\int T_{m,N} R^{-1}(f)(t)\, T_{m',N} R^{-1}(g)(t)\, d\nu_{m,m'}(t) = \int T_{m,h}(f)(t)\, T_{m',h}(g)(t)\, d\mu_{m,m'}(t).$$

As $B(f, g) = B'(R^{-1}(f), R^{-1}(g))$, Condition (b) follows from (2.20).

Now, we prove (b) → (c). Indeed, given $0 \leqslant l \leqslant N-1$ and $k \geqslant 0$, we put $a_{l,k} = \int t^l h^k(t)\, d\mu_{0,l}(t)$. Since $\{t^m h^k, k \geqslant 0, 0 \leqslant m \leqslant N-1\}$ is a basis of $\mathscr{P}$, it follows that there exists a function μ_0 such that

$$a_{l,k} = \int t^l h^k(t)\, d\mu_0(t).$$

Hence, we get

$$\sum_{l=0}^{N-1} \int (T_{0,h}(f)(t)\, T_{l,h}(g)(t) + T_{l,h}(f)(t)\, T_{0,h}(g)(t))\, d\mu_{0,l}$$
$$= \sum_{l=0}^{N-1} \int (T_{0,h}(f)(t)\, T_{l,h}(g)(t) + T_{l,h}(f)(t)\, T_{0,h}(g)(t))\, d\mu_0.$$

If f is a polynomial then $f = \sum_{m=0}^{N-1} T_{m,h} f$. So, we can write B as

$$B(f, g) = \int f(t)\, g(t)\, d\mu_0$$
$$+ \sum_{1 \leqslant m \leqslant m' \leqslant N-1} \int T_{m,h}(f)(t)\, T_{m',h}(g)(t)(d\mu_{m,m'}(t) - d\mu_0(t)),$$

and so, Condition (c) follows.

Since $T_{m,h}(hf)(t) = h(t)\, T_{m,h}(f)(t)$, for $0 \leqslant m \leqslant N-1$, (b) → (a) follows easily. ∎

Note that a h-generalized $(2N+1)$-term recurrence relation can be defined for a polynomial h of degree N, changing the polynomial t^N by h in the formula (2.8)((2.9)). Theorem 4 can then be extended for this h-generalized $(2N+1)$-term recurrence relation.

3. Inner Products of Discrete Sobolev Type

During the past few years, several papers have been written about orthonormal polynomials with respect to an inner product of the form (see [IKNS1, IKNS 2, BM1, BM2, MR, K, MV], ...)

$$B(f, g) = \int f(t)\, g(t)\, d\mu + \sum_{l=1}^{K} \int f^{(l)}(t)\, g^{(l)}(t)\, d\mu_l, \tag{3.1}$$

especially when μ_l are discrete measures (see [BM1, BM2, MR, K, MV], ...).

Here, we should refer to the recent preprint [ELMMR], because some of the results which have been proved there, are strongly related to the results we will prove below. In [ELMMR], necessary and sufficient conditions are given on the positive measures μ_l, $l=1, \ldots, K$, in order that there exists a real-valued polynomial h symmetric with respect to the inner product defined by (3.1) (that is, $B(hp, q)=B(p, hq)$ for all polynomials p, q).

Indeed, the positive measures μ_l must be purely atomic with a finite number of mass points, that is, a finite combinations of Dirac deltas.

In this section, we give necessary and sufficient conditions on a real bilinear form B (not necessarily as (3.1)), in terms of symmetric operators, in order that B is defined by (3.1) when the measures μ_l are Dirac's deltas at the same point, or when $\mu_l=0$ for $l \neq 1$, and μ_1 is a finite combination of Dirac deltas. However, using the same technique, a characterization could be given for the general case when the measures μ_l are discrete (that is, finite combinations of Dirac deltas).

Before turning to prove the main results in this Section, we give some examples:

EXAMPLES. (I) Note that if μ and μ_l are positive measures, then B defined as in (3.1) is an inner product. However, the converse is not true. An example of this fact is presented below.

We take $K=2$. Let $(\alpha_n)_n$ be the sequence defined by

$$\alpha_n = \begin{cases} 3, & \text{for } n=0, \\ -\dfrac{3}{2}, & \text{for } n=1, \\ \dfrac{1}{n+1}, & \text{for } n \geqslant 2. \end{cases}$$

Put μ for a function such that $\int t^n \, d\mu(t) = \alpha_n$ for $n \geqslant 0$. Now, we define B as follows:

$$B(f, g) = \int f(t)\, g(t)\, d\mu + 2f'(0)\, g'(0).$$

Since $(\alpha_n)_n$ is not a positive definite sequence, it is clear that B cannot be written as

$$\int f(t)\, g(t)\, dv(t) + \int f'(t)\, g'(t)\, dv_1(t) \tag{3.2}$$

for positive measures v, v_1. In order to prove that B is an inner product, we observe that $\alpha_0 = 1+2$, $\alpha_1 = \frac{1}{2} - 2$ and write B as

$$B(f, g) = \int f(t)\, g(t)\, d\chi_{[0,1]} + 2f(0)\, g(0)$$
$$- 2(f(0)\, g'(0) + f'(0)\, g(0)) + 2f'(0)\, g'(0).$$

The result follows because $B(f, f) = \int f^2(t)\, d\chi_{[0,1]} + 2(f(0) - f'(0))^2$.

Other examples can be given by using Lemma 1.

(II) Just be using Lemma 1, we are going to show that the above mentioned result in [ELMMR] is sharp in the following way: if we remove the positivity of the measures μ_l which appear in (3.1), then the result does not hold. Indeed, we show an example of an inner product defined by

$$B(p, q) = \int p(t)\, q(t)\, d\mu_1(t) + \int p'(t)\, q'(t)\, d\mu_2(t),$$

where μ_1 is a positive measure and μ_2 is a function, such that, the operator t^3 is symmetric for B and B can not be written as

$$B(p, q) = \int p(t)\, q(t)\, dv_1(t) + \int p'(t)\, q'(t)\, dv_2(t)$$

for any positive measure v_1 and any finite combinations of Dirac deltas v_2.

To show this example, let us consider the following matrix $T = (t_{i,j})_{i,j=0}^{\infty}$ defined by

$$t_{i,j} = \begin{cases} 2, & \text{if } \ i=1, j=2, \\ 2, & \text{if } \ i=2, j=1, \\ 0, & \text{otherwise.} \end{cases}$$

From Lemma 1, it follows that there exists a sequence of real numbers $(a_n)_n$ such that the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ defined by (2.17), (2.18) are positive. Hence, $(a_n)_n$ is a positive definite sequence. Let μ, v be a positive measure such that $\int t^n\, d\mu(t) = a_n$ and a function such that

$$\int t^n\, dv(t) = \begin{cases} 1, & \text{if } \ n=1, \\ 0, & \text{otherwise,} \end{cases}$$

respectively. We define the real symmetric bilinear form B by

$$B(p, q) = \int p(t)\, q(t)\, d\mu(t) + \int p'(t)\, q'(t)\, dv(t).$$

It is clear that B can be written as

$$B(p, q) = \int p(t)\, q(t)\, d\mu(t) + 2(p'(0)\, q''(0) + p''(0)\, q'(0)).$$

From here, it follows that $B(t^3 p(t), q(t)) = B(p(t), t^3 q(t))$, that is t^3 is symmetric for B, and that this bilinear form cannot be written as

$$B(p, q) = \int p(t)\, q(t)\, d\nu_1(t) + \int p'(t)\, q'(t)\, d\nu_2(t)$$

for any positive measure ν_1, and any finite combination of Dirac deltas ν_2.

But from the definition of B, we get that $B(t^i, t^j) = a_{i+j} + t_{i,j}$, and since the numbers β_n are positive, we have that B is an inner product.

Now, we study the special case when the measures μ_l, which appear in (3.1), are Dirac deltas at the same point, say at 0. We give the characterization as a corollary of the following:

THEOREM 6. *Let B be a real symmetric bilinear form defined on $\mathcal{P}$ and N a non-negative integer. Then the following are equivalent:*

(a) *The operator t^N is symmetric for B, and $B(t^N f, tg) = B(tf, t^N g)$ if f, g are polynomials.*

(b) *There exists a function μ and constants $M_{k,m}$, $1 \leqslant k, m \leqslant N-1$, with $M_{k,m} = M_{m,k}$ such that*

$$B(f, g) = \int f(t)\, g(t)\, d\mu(t) + \sum_{k,m=1}^{N-1} M_{k,m} f^{(k)}(0)\, g^{(m)}(0).$$

Proof. We prove (a) → (b), the converse is straightforward. Consider the sequence $\alpha_n = B(1, t^n)$. We put

$$M'_{k,m} = B(t^k, t^m) - \alpha_{k+m} \tag{3.3}$$

if $1 \leqslant k, m \leqslant N-1$. We consider a function μ such that $\int t^n\, d\mu(t) = \alpha_n$. Then from the hypothesis, we get, if m' or k' are different from 0 and if $1 \leqslant k$, $m \leqslant N-1$:

$$B(t^{m'N+m}, t^{k'N+k}) = \alpha_{m'N+k'N+m+k},$$
$$B(t^m, t^k) = M'_{k,m} + \alpha_{k+m}.$$

Now, if $f=\sum_i a_i t^i$ and $g=\sum_j b_j t^j$, we get

$$\begin{aligned}
B(f, g) &= \sum_{i,j} a_i b_j B(t^i, t^j) \\
&= \sum_{0\leqslant i,j\leqslant N-1} a_i b_j B(t^i, t^j) + \sum_{\substack{0\leqslant j\\ N\leqslant i}} a_i b_j B(t^i, t^j) \\
&\quad + \sum_{\substack{0\leqslant i\\ N\leqslant j}} a_i b_j B(t^i, t^j) - \sum_{\substack{N\leqslant i\\ N\leqslant j}} a_i b_j B(t^i, t^j) \\
&= \sum_{0\leqslant i,j\leqslant N-1} a_i b_j \alpha_{i+j} + \sum_{1\leqslant i,j\leqslant N-1} a_i b_j M'_{i,j} \\
&\quad + \sum_{\substack{0\leqslant j\\ N\leqslant i}} a_i b_j \alpha_{i+j} + \sum_{\substack{0\leqslant i\\ N\leqslant j}} a_i b_j \alpha_{i+j} - \sum_{\substack{N\leqslant i\\ N\leqslant j}} a_i b_j \alpha_{i+j} \\
&= \sum_{i,j} a_i b_j \alpha_{i+j} + \sum_{1\leqslant i,j\leqslant N-1} a_i b_j M'_{i,j}.
\end{aligned}$$

Since $i!\, a_i = f^{(i)}(0)$, we put $M_{k,m} = (k!\, m!)^{-1} M'_{k,m}$, to find

$$B(f, g) = \int f(t)\, g(t)\, d\mu(t) + \sum_{k,m=1}^{N-1} M_{k,m} f^{(k)}(0)\, g^{(m)}(0).$$

COROLLARY 7. *Let B be a real symmetric bilinear form defined on $\mathscr{P}$ and N a non-negative integer. Then the following are equivalent*:

(a) *The operator t^N is symmetric for B; $B(t^N f, tg) = B(tf, t^N g)$ if f, g are polynomials, and $B(t^k, t^m) = B(1, t^{k+m})$ when $1 \leqslant k, m \leqslant N-1$ and $k \neq m$.*

(b) *There exists a function μ and constants M_k, $1 \leqslant k \leqslant N-1$, such that*

$$B(f, g) = \int f(t)\, g(t)\, d\mu(t) + \sum_{k=1}^{N-1} M_k f^{(k)}(0)\, g^{(k)}(0).$$

Proof. From $B(t^k, t^m) = B(1, t^{k+m})$ when $1 \leqslant k$, $m \leqslant N-1$ and $k \neq m$, if follows that the constants $M'_{k,m}$ defined in (3.3) are equal to 0 when $k \neq m$. ∎

Note that a similar characterization can be given when the measures μ_l are Dirac deltas at the same point $c \in \mathbf{R}$. In this case, the polynomial t^N which appears in Theorem 6 and Corollary 7, must be changed to the polynomial $(t-c)^N$.

Now, we are going to study the real bilinear forms defined in (3.1), when $\mu_l = 0$ for $l \neq 1$ and

$$\mu_1 = \sum_{l=1}^{K} M_l \delta_{a_l}$$

(we put δ_{a_l} for the Dirac delta with mass at the point a_l. This functional is also denoted by $\delta(x - a_l)$).

The following two Lemmas will be useful to give the characterization of these bilinear forms, although they are interesting in themselves:

LEMMA 2. *Let B be a real symmetric bilinear form defined on $\mathscr{P}$ and K a non-negative integer. Consider a finite sequence of real numbers $(a_l)_{l=1}^{K}$ and non-negative integers n_l, $1 \leqslant l \leqslant K$. Let h be the polynomial $h(t) = (t-a_1)^{n_1} \cdots (t-a_K)^{n_K}$ and $N = n_1 + \cdots + n_K = dgr(h)$. Then the following are equivalent:*

(a) *If f, g are polynomials, then $B(hf, g) = 0$.*

(b) *There exist constants $M_{i,j,l,l'}$ with $0 \leqslant i \leqslant n_l - 1$, $0 \leqslant j \leqslant n_{l'} - 1$, $1 \leqslant l, l' \leqslant K$ and $M_{i,j,l,l'} = M_{j,i,l',l}$, such that the bilinear form B is defined by*

$$B(f, g) = \sum_{l,l'=1}^{K} \sum_{i=0}^{n_l - 1} \sum_{j=0}^{n_{l'} - 1} M_{i,j,l,l'} f^{(i)}(a_l)\, g^{(j)}(a_{l'}).$$

Proof. (b) → (a) is straightforward. Hence, we only prove (a) → (b). We consider the following basis of $\mathscr{P}$:

$$\mathscr{B}_1 = \{1, (t-a_1)^{n_1} (t-a_2)^{n_2} \cdots (t-a_l)^i h^j : l = 1, ..., K, 1 \leqslant i \leqslant n_l, j \geqslant 0\}.$$

Put $w_{i,l} = (t-a_1)^{n_1} (t-a_2)^{n_2} \cdots (t-a_l)^i$ where, $1 \leqslant i \leqslant n_l$ for $l = 2, ..., K-1$, $0 \leqslant i \leqslant n_1$ for $l = 1$ and $1 \leqslant i \leqslant n_K - 1$ for $l = K$.

Since any polynomial p can be written as $p = qh + r$, where q and r are polynomials and $\text{dgr}(r) \leqslant N - 1$, it follows that if two bilinear forms B, B_1 satisfy condition a) and $B(f, g) = B_1(f, g)$ when $\text{dgr}(f)$, $\text{dgr}(g) \leqslant N - 1$, then $B = B_1$. So, given B satisfying condition (a), it is sufficient to prove that there exist constants $M_{i,j,l,l'}$ with $0 \leqslant i \leqslant n_l - 1$, $0 \leqslant j \leqslant n_{l'} - 1$, $1 \leqslant l$, $l' \leqslant K$ and $M_{i,j,l,l'} = M_{j,i,l',l}$ such that the bilinear form B_1 defined

$$B_1(f, g) = \sum_{l,l'=1}^{K} \sum_{i=0}^{n_l - 1} \sum_{j=0}^{n_{l'} - 1} M_{i,j,l,l'} f^{(i)}(a_l)\, g^{(j)}(a_{l'})$$

is equal to B, when $\text{dgr}(f)$, $\text{dgr}(g) \leqslant N - 1$. Since the polynomials $w_{i,l}$ when $1 \leqslant i \leqslant n_l$ for $l = 2, ..., K-1$, $0 \leqslant i \leqslant n_1$ for $l = 1$ and $1 \leqslant i \leqslant n_K - 1$ for $l = K$ are a basis of

$$\mathscr{P}_{N-1} = \{p : p \text{ is a polynomial of degree } \leqslant N - 1\},$$

the previous fact is equivalent with the existence of constants $M_{i,j,l,l'}$ with $M_{i,j,l,l'} = M_{j,i,l',l}$ such that

$$B(w_{i,l}, w_{j,l'}) = B_1(w_{i,l}, w_{j,l'}) \tag{3.4}$$

when $1 \leqslant i \leqslant n_l$, $1 \leqslant j \leqslant n_{l'}$ for $l, l' = 2, ..., K-1$, $0 \leqslant i, j \leqslant n_1$ for $l, l' = 1$ and $1 \leqslant i, j \leqslant n_K - 1$ for $l, l' = K$.

Now, note that (3.4) gives a system of N^2 linear equations with unknowns $M_{i,j,l,l'}$.

We write these equations (3.4) in the following order:

$$\begin{aligned}
&(w_{n_K-1,K}, w_{n_K-1,K}) \to (w_{n_K-1,K}, w_{n_K-2,K}) \to \cdots \to (w_{n_K-1,K}, 1)\\
&\quad \to (w_{n_K-2,K}, w_{n_K-1,K}) \to (w_{n_K-2,K}, w_{n_K-2,K}) \to \cdots \to (w_{n_K-2,K}, 1)\\
&\quad \to \cdots\\
&\quad \to (1, w_{n_K-1,K}) \to (1, w_{n_K-2,K}) \to \cdots \to (1, 1).
\end{aligned}$$

Thus, (3.4) actually gives a triangular system of N^2 linear equations with unknowns $M_{i,j,l,l'}$. And so, these constants are determinated by (3.4).

Since B is symmetric and from the definition of B_1, the condition $M_{i,j,l,l'} = M_{j,i,l',l}$ follows. Hence, the Lemma is proved. ∎

LEMMA 3. *Let B be a real symmetric bilinear form defined on $\mathscr{P}$, and K a non-negative integer. Consider a finite sequence of real numbers $(a_l)_{l=1}^K$ and non-negative integers n_l, $1 \leqslant l \leqslant K$. Let h be the polynomial $h(t) = (t-a_1)^{n_1} \cdots (t-a_K)^{n_K}$ and $N = n_1 + \cdots + n_K = dgr(h)$. Then the following are equivalent:*

(a) *There exists a function μ, constants $M_{i,j,l,l'}$ with $0 \leqslant i \leqslant n_l - 1$, $0 \leqslant j \leqslant n_{l'} - 1$, $1 \leqslant l$, $l' \leqslant N-1$* and *$M_{i,j,l,l'} = M_{j,i,l',l}$ such that the bilinear form B is defined as*

$$B(f, g) = \int f(t)\, g(t)\, d\mu(t) + \sum_{l,l'=1}^{K} \sum_{i=0}^{n_l-1} \sum_{j=0}^{n_{l'}-1} M_{i,j,l,l'} f^{(i)}(a_l)\, g^{(j)}(a_{l'}).$$

(b) *The operator h is symmetric for B and $B(hf, tg) = B(tf, hg)$ for all polynomials f, g.*

Proof. We prove (b) → (a).

Let $\mathscr{B}_h = \{v_{m+iN} = t^m h^i : 0 \leqslant m \leqslant N-1, i \geqslant 0\}$ be the basis of $\mathscr{P}$, defined in the previous Section. Given $0 \leqslant m \leqslant N-1$ and $i \geqslant 0$, consider the sequence $a_{m,i} = B(1, t^m h^i)$. Since $\mathscr{B}_h$ is a basis of $\mathscr{P}$, there exists a function

ν such that $\int t^m h^i \, d\nu(t) = a_{m,i}$. Now, we define the real symmetric bilinear forms B_1 and B_2 by

$$B_1(x^m h^i, x^l h^j) = \begin{cases} B(x^m h^i, x^l h^j), & \text{if } i \geq 1 \text{ or } j \geq 1, \\ \int x^m x^l \, d\nu(t), & \text{if } i = 0 \text{ and } j = 0. \end{cases}$$

and $B_2(f, g) = B(f, g) - B_1(f, g)$.

We proceed in several steps:

First step. For all polynomials f, g, we have $B_1(hf, g) = B(hf, g)$, and so $B_2(hf, g) = 0$. The step follows because by definition of B_1, we get

$$B_1(hv_{m+iN}, g) = B_1(v_{m+(i+1)N}, g) = B(v_{m+(i+1)N}, g) = B(hv_{m+iN}, g)$$

when v_{m+iN} belongs to the basis $\mathscr{B}_h$.

Second step. For all polynomials f, g, we have $B_1(tf, g) = B_1(f, tg)$.

It will be enough to prove this for f, g belonging to the basis $\mathscr{B}_h$. If we consider $f = t^m h^i$ and $g = t^l h^j$, with $i \geq 1$ or $j \geq 1$, then

$$\begin{aligned} B_1(tt^m h^i, t^l h^j) &= B(tt^m h^i, t^l h^j) = B(tt^m, t^l h^{i+j}) = B(t^m h, tt^l h^{i+j-1}) \\ &= B(t^m h^i, tt^l h^j) = B_1(t^m h^i, tt^l h^j), \end{aligned}$$

where we have used the first step and Condition (b).

Note that the second step follows if we prove that $B_1(t^N, t^m) = \int t^N t^m \, d\nu(t)$ when $1 \leq m \leq N-1$. But, if we put $t^N = \sum_{l=0}^{N-1} b_l t^l + b_N h$, we get

$$\begin{aligned} B_1(t^N, t^m) &= \sum_{l=0}^{N-1} b_l B_1(t^l, t^m) + b_N B_1(h, t^m) \\ &= \sum_{l=0}^{N-1} b_l \int t^l t^m \, d\nu(t) + b_N B(h, t^m) \\ &= \sum_{l=0}^{N-1} b_l \int t^l t^m \, d\nu(t) + b_N B(1, t^m h) \\ &= \sum_{l=0}^{N-1} b_l \int t^l t^m \, d\nu(t) + b_N \int t^m h \, d\nu(t) \\ &= \int \left(\sum_{l=0}^{N-1} b_l t^l + b_N h \right) t^m \, d\nu(t) = \int t^N t^m \, d\nu(t). \end{aligned}$$

Hence, the second step is proved.

Now, from Theorem 3 for $N = 1$, a function μ such that $B_1(f, g) = \int f(t) \, g(t) \, d\mu$ is obtained. It is enough to apply Lemma 2 in order to finish the proof. ∎

Given a finite sequence of real numbers $(a_l)_{l=1}^{K}$, consider the following polynomials

$$q_{i,m}(t)=(t-a_1)^2\cdots(t-a_m)^{2-i}\cdots(t-a_K)^2 \tag{3.5}$$

for $i=1, 2$ and $1\leqslant m\leqslant K$.

THEOREM 8. *Let B be a symmetric bilinear form defined on $\mathscr{P}$ and K a non-negative integer. Consider a finite sequence of real numbers $(a_l)_{l=1}^{K}$. Let h be the polynomial $h(t)=(t-a_1)^2\cdots(t-a_K)^2$. Then the following are equivalent:*

(a) *There exists a function μ and constants M_l with $1\leqslant l\leqslant K$, such that the bilinear form B is defined by*

$$B(f, g)=\int f(t)\, g(t)\, d\mu(t)+\sum_{l=1}^{K} M_l f'(a_l)\, g'(a_l).$$

(b) *The operator h is symmetric for B; $B(hf, tg)=B(tf, hg)$ for all polynomials f, g, and*

$$B(q_{i,m}, q_{j,m'})=B(1, q_{i,m}q_{j,m'}) \tag{3.6}$$

for $i, j=1, 2$, $1\leqslant m, m'\leqslant K$ and $m\neq m'$, where $q_{i,m}$ are the polynomials defined in (3.5).

Proof. Again, we only prove (b) → (a).

From Condition (b) and Lemma 3, we obtain the following expression for B:

$$B(f, g)=\int f(t)\, g(t)\, d\mu(t)+\sum_{l,l'=1}^{K}\ \sum_{i,j=0}^{1} M_{i,j,l,l'} f^{(i)}(a_l)\, g^{(j)}(a_{l'}). \tag{3.7}$$

From the choice of the polynomials $q_{i,m}$, we obtain

$$(q_{1,m})^{(i)}(a_l)\neq 0 \qquad \text{if and only if} \quad i=1 \text{ and } l=m, \tag{3.8}$$

and

$$(q_{2,m})^{(i)}(a_l)\neq 0 \qquad \text{if and only if} \quad i=0, 1 \text{ and } l=m. \tag{3.9}$$

Now, from (3.6), (3.7), and (3.8) we get $M_{1,1,l,l'}=0$ if $l\neq l'$, and from (3.6), (3.7), and (3.9) we get $M_{0,1,l,l'}=0$ if $l\neq l'$. Hence, from the expression (3.7), we get

$$B(f, g)=\int f(t)\, g(t)\, d\mu(t)+\sum_{l=1}^{K} M_{1,1,l,l} f'(a_l)\, g'(a_l)$$
$$+\sum_{l=1}^{K} M_{0,1,l,l}(f(a_l)\, g'(a_l)+f'(a_l)\, g(a_l)). \tag{3.10}$$

Putting $\nu=\sum_{l=1}^{K} M_{0,1,l,l}\delta_{a_l}$, we have

$$\sum_{l=1}^{K} M_{0,1,l,l}(f(a_l)\, g'(a_l)+f'(a_l)\, g(a_l))=\int (fg)'(t)\, d\nu(t).$$

The operator t is symmetric for the real symmetric bilinear form $B_1(f, g)=\int (fg)'(t)\, d\nu$, hence there exists a function ρ such that $B_1(f, g)=\int f(t)\, g(t)\, d\rho$. If we put this expression for B_1 in (3.10), we get

$$B(f, g)=\int f(t)\, g(t)(d\mu(t)+d\rho(t))+\sum_{l=1}^{K} M_{1,1,l,l} f'(a_l)\, g'(a_l),$$

and so, the theorem is proved. ∎

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